ON THE MOTION AND STABILITY OF A GYROSCOPE ON GIMBALS IN THE NEWTONIAN CENTRAL FORCE FIELD

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In [1-5] the authors have investigated the motion and also the stability of a certain stationary motion of symmetric gyroscopes, with the axis of the outer gimbal ring vertical, in the uniform gravitational force field. This problem is closely related to similar problems arising in the motion of a rigid body about a fixed point in the case of Lagrange.

The author of [6] has investigated a rigid body moving about a fixed point, assuming that the dimensions of the body are small compared with the distance from the fixed point to the center of attraction. His case, similar to the case of Lagrange, has been reduced to quadratures. The necessary conditions for stability of permanent rotations for the above case have been presented in [7].

The problem investigated here is described in the title. It is assumed that the direction of the axis of the outer gimbal ring coincides with the direction of the line from the attraction center to the point of intersection of the gimbal axes. This assumption permits, as might be expected, an analogy with the case of Lagrange. The dimensions of the body are assumed, as in [6], to be relatively small. The integration of the equations of motion is reduced to quadratures. When investigating the stability of stationary solutions (regular precession and "vertical rotation") the method of Chetaev has been applied.

1. Consider a gyroscope on gimbals, and introduce two rectangular coordinate systems $Ox_1y_1z_1$ and $Oxyz$ with the origin at the point of intersection of the gimbal axes. The system $Ox_1y_1z_1$ is fixed, the Oz_1-axis is along the axis of the outer gimbal ring. The $Oxyz$-system moves with the inner ring (the housing), the Ox- and Oz-axes are, respectively, along the axis of rotation of the inner ring and along the axis of symmetry of the gyroscope.

The orientation of the gyroscope with respect to the fixed coordinate system is determined by the Eulerian angles: ψ the precession angle, θ the nutation angle and the angle of spin, ϕ the rotation angle of the gyroscope with respect to the system $Oxyz$.

Let I be the moment of inertia of the outer ring about the Oz_1-axis, A^o, B^o; C^o be, respectively, the moments of inertia of the housing about the x-, y- and z-axes, A, $B = C$ be, respectively, the moments of inertia of the gyroscope about the x-, y- and z-axes. The axes x, y and z are the principal axes of the ellipsoid of inertia of the case and of the gyroscope as well.

Using the above notation the kinetic energy T of the system can be written as

$$2T = (A + A^\circ)\dot{\theta}^2 + [I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\dot{\psi}^2 + C(\dot{\varphi} + \dot{\psi}\cos\theta)^2$$

We assume that the attracting center O_1 is on the negative branch of the Oz_1-axis and the distance between the attraction center O_1 and the origin O, $O_1O = R$ is very large as compared with the dimensions of the gyroscope and the gimbal rings. Consequently, in the expressions for the force components F_x, F_y, F_z acting on the mass element dm, the small quantities of the second order and higher can be neglected and these force components can be written as

$$F_{x_1} = -\frac{gdm}{R}x_1, \qquad F_{y_1} = -\frac{gdm}{R}y_1, \qquad F_{z_1} = -gdm + \frac{2gdm}{R}z_1$$

Here g is the gravitational acceleration at the distance R from the attraction center, and x_1, y_1, z_1 are the coordinates of the mass element dm.

The center of mass of the system consisting of the housing and the gyroscope is on the z-axis and $l \geqslant 0$ is its z-coordinate; M is the mass of the housing and the gyroscope. Assuming that the acting forces are only those due to gravity, we can write the expression for the differential of the force function as

$$dU = -\frac{g}{R}\sum[x_1dx_1 + y_1dy_1 + (R - 2z_1)dz_1]\,dm$$

hence

$$U = -\frac{g}{2R}\sum(x_1^2 + y_1^2 - 2z_1^2 + 2Rz_1)\,dm$$

After passing from the fixed axes x_1, y_1, z_1 to the principal axes x, y, z, performing certain simple calculations, and rejecting constant terms we obtain finally

$$U = -Mgl\cos\theta + \frac{3g}{2R}(A + B^\circ - C - C^\circ)\cos^2\theta \tag{1.1}$$

If the attraction center moves away from the point O to infinity ($R \to \infty$) then, in the limit, the second term vanishes and what remains on the left-hand side of (1.1) is the well-known expression for the force function resulting from the uniform gravitational force field, for a gyroscope with the axis of the outer gimbal ring vertical. It is of interest to note that a similar coincidence occurs also when the moments of inertia of the gyroscope and the housing satisfy the condition

$$A + B^\circ - C - C^\circ = 0 \tag{1.2}$$

If this occurs, the motion of a gyroscope is the same as the motion of a gyroscope with the axis of the outer gimbal ring vertical in the uniform gravitational force field. Since this latter case has been investigated in detail in [1-5] it will be excluded from our investigations, and the condition (1.2) will not be satisfied in our case.

Since the coordinates θ, ψ, ϕ are independent and holonomic, the equations of motion can be written in form of the Lagrange equations of the second kind. We have then

$$\begin{gathered}(A + A^\circ)\ddot\theta - (A + B^\circ - C^\circ)\dot\psi^2\sin\theta\cos\theta + C(\dot\varphi + \dot\psi\cos\theta)\dot\psi\sin\theta - \\ - Mgl\sin\theta + \frac{3g}{R}(A + B^\circ - C - C^\circ)\sin\theta\cos\theta = 0 \\ \frac{d}{dt}\{[I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\dot\psi + C(\dot\varphi + \dot\psi\cos\theta)\cos\theta\} = 0 \\ \frac{d}{dt}[C(\dot\varphi + \dot\psi\cos\theta)] = 0\end{gathered} \tag{1.3}$$

If the masses of the gimbal rings are neglected, then Equations (1.3) reduce to the equations given in [6] for a case analogous to the Lagrange case.

The equations of motion permit us to establish the following first integrals:

$$\begin{gathered}(A + A^\circ)\dot\theta^2 + [I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\dot\psi^2 + C(\dot\varphi + \dot\psi\cos\theta)^2 + \\ + 2Mgl\cos\theta - \frac{3g}{R}(A + B^\circ - C - C^\circ)\cos^2\theta = h \\ [I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\dot\psi + C(\dot\varphi + \dot\varphi\cos\theta)\cos\theta = k \\ \dot\varphi + \dot\psi\cos\theta = r\end{gathered} \tag{1.4}$$

where the first one is the kinetic energy integral, and the remaining two correspond to the cyclic coordinates ψ and ϕ; h and r are the integration constants.

2. We shall now reduce the integration of the equations of motion (1.3) to quadratures. From (1.4) we have the following system of differential equations for the Eulerian angles:

$$\frac{d\psi}{dt} = \frac{\beta - bru}{\varepsilon - eu^2}, \qquad \frac{d\varphi}{dt} = r - \frac{\beta - bru}{\varepsilon - eu^2}u \tag{2.1}$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{(\alpha - au + a_1u^2)(\varepsilon - eu^2) - (\beta - bru)^2}{\varepsilon - eu^2}$$

Here we use the following notation:

$$u = \cos\theta$$

$$\alpha = \frac{h - Cr^2}{A + A^\circ}, \qquad a = \frac{2Mgl}{A + A^\circ} \geqslant 0, \qquad a_1 = \frac{3g(A + B^\circ - C - C^\circ)}{R(A + A^\circ)}$$

$$\varepsilon = \frac{I + A + B^\circ}{A + A^\circ} > 0, \qquad e = \frac{A + B^\circ - C^\circ}{A + A^\circ}, \qquad \beta = \frac{k}{A + A^\circ}, \qquad b = \frac{C}{A + A^\circ} > 0$$

The integration of the system (2.1) starts from the last equation, from which, after taking into account that $\dot{u} = -\dot{\theta}\sin\theta$, we obtain

$$t - t_0 = \int_{u_0}^{u} \frac{(\varepsilon - eu^2)\,du}{\sqrt{[(\alpha - au + a_1u^2)(\varepsilon - eu^2) - (\beta - bru)^2](\varepsilon - eu^2)(1 - u^2)}}$$

After differentiating this hyperbolic integral and solving for u, the solution of the first two equations of the system (2.1) determining the angles ψ and ϕ reduces to quadratures. Let

$$f(u) = (\alpha - au + a_1u^2)(\varepsilon - eu^2) - (\beta - bru)^2$$

and in order to be specific let us limit ourselves to the case when $e > 0$ and $a_1 < 0$. We propose to show that when these conditions are satisfied then the stability of the gyroscope's regular precession at constant precessional velocity (of an arbitrary magnitude) is insured. In this case the polynomial $f(u)$ has (in the mechanical problem) four real roots u', u_1, u_2, u'' contained, respectively, in the following intervals:

$$u' < -\sqrt{\frac{\varepsilon}{e}}, \qquad -\sqrt{\frac{\varepsilon}{e}} < u_1 \leqslant u_0, \qquad u_0 \leqslant u_2 < \sqrt{\frac{\varepsilon}{e}}, \qquad u'' > \sqrt{\frac{\varepsilon}{e}}$$

The roots u' and u'' are numerically larger than unity, therefore the quantity u, beginning from the value u_0, must remain all the time in the interval between those of the two neighboring points -1, -1, u_1, u_2 where the point u_0 is located.

We shall consider the motion determined by the initial conditions

$$\theta_0 \neq 0, \qquad \dot\theta_0 = 0, \qquad \dot\psi_0 = 0, \qquad r = r_0$$

where the constant r_0 is numerically large.

These initial conditions lead in the case of Lagrange to a pseudo-regular precession.

From (2.1) we have

$$\beta - br_0u_0 = 0, \qquad \alpha - au_0 + a_1u_0^2 = 0$$

hence

$$f(u) = (u_0 - u)\{[a - a_1(u_0 + u)](\varepsilon - eu^2) - b^2r_0^2(u_0 - u)\}$$

From this

$$u_0 - u_1 = \frac{[a - a_1(u_0 + u_1)](\varepsilon - eu_1^2)}{b^2r_0^2}$$

Examination of the sign on the right-hand side shows that if we take into account the values assumed for e, a_1 and with $u_0 \geqslant 0$ $(0 \leqslant \theta \leqslant (1/2)\pi)$ then $u_1 < u_0$. Consequently, at large values of r_0 the value of u can vary only inside the interval $[u_1, u_2 = u_0]$, and this interval is decreasing as r_0 increases.

3. The equations of motion (1.3) permit the following particular solution:

$$\theta = \theta_0, \qquad \dot\theta = 0, \qquad \dot\psi = \dot\psi_0, \qquad r = r_0 \tag{3.1}$$

when the constants θ_0, $\dot\psi_0$, r_0 satisfy the condition

$$\Big[(A + B^\circ - C^\circ)\dot\psi_0^2\cos\theta_0 - Cr_0\dot\psi_0 + Mgl - \\ - \frac{3g}{R}(A + B^\circ - C - C^\circ)\cos\theta_0\Big]\sin\theta_0 = 0 \tag{3.2}$$

We shall consider first the motion (3.1) when $\theta \neq 0, \pi$. In this case Equations (3.1) represent the regular precession of the gyroscope, and condition (3.2) is satisfied because the expression inside the brackets equals zero:

$$(A + B^\circ - C^\circ)\dot\psi_0^2\cos\theta_0 - Cr_0\dot\psi_0 + Mgl - \frac{3g}{R}(A + B^\circ - C - C^\circ)\cos\theta_0 = 0 \tag{3.3}$$

This quadratic equation in $\dot\psi_0$ will have real roots if the condition

$$C^2r_0^2 - 4(A + B^\circ - C^\circ)\Big[Mgl - \frac{3g}{R}(A + B^\circ - C - C^\circ)\cos\theta_0\Big]\cos\theta_0 \geqslant 0$$

or

$$C^2\dot\psi_0^2 - 4(A + B^\circ - C - C^\circ)\left[Mgl - \frac{3g}{R}(A + B^\circ - C - C^\circ)\cos\theta_0\right]\cos\theta_0 \geqslant 0$$

is satisfied.

Let us substitute for the perturbed motion

$$\theta = \theta_0 + \eta, \qquad \dot\theta = \dot\eta = \xi_1, \qquad \dot\psi = \dot\psi_0 + \xi_2, \qquad r = r_0 + \xi_3$$

The integrals (1.4) of the unperturbed motion have the following analogs in the case of the perturbed motion:

$$V_1 = (A + A^\circ)\xi_1^2 + [(A + B^\circ - C^\circ)\dot\psi_0^2(n^2 - m^2) - Mg\,ln +$$
$$+ \frac{3g}{R}(A + B^\circ - C - C^\circ)(n^2 - m^2)]\eta^2 + [I + C^\circ + (A + B^\circ - C^\circ)m^2] \times$$
$$\times (\xi_2^2 + 2\dot\psi_0\xi_2) + 4(A + B^\circ - C^\circ)\dot\psi_0 mn\eta\xi_2 + C(\xi_3^2 + 2r_0\xi_3) +$$
$$+ 2\left[(A + B^\circ - C^\circ)\dot\psi_0^2 n - Mgl + \frac{3g}{R}(A + B^\circ - C - C^\circ)n\right]m\eta + \ldots = \text{const}$$

$$V_2 = [(A + B^\circ - C^\circ)\dot\psi_0(n^2 - m^2) - \tfrac{1}{2}Cr_0 n]\eta^2 +$$
$$+ [2(A + B^\circ - C^\circ)\dot\psi_0 n - Cr_0]m\eta + [I + C^\circ + (A + B^\circ - C^\circ)m^2]\xi_2 +$$
$$+ 2(A + B^\circ - C^\circ)mn\eta\xi_2 + C(n - m\eta)\xi_3 + \ldots = \text{const}$$

$$V_3 = \xi_3 = \text{const}$$

where $m = \sin\theta_0$, $n = \cos\theta_0$, and the dots indicate the neglected terms of higher order. We shall consider the following integral of the perturbed motion:

$$V = V_1 - 2\dot\psi_0 V_2 + 2C(\dot\psi_0 n - r_0)V_3 +$$
$$+ \frac{C^2\dot\psi_0^2}{(A + B^\circ - C^\circ)\dot\psi_0^2 - (3g/R)(A + B^\circ - C - C^\circ)}V_3^2 = (A + A^\circ)\xi_1^2 +$$
$$+ [I + C^\circ + (A + B^\circ - C^\circ)m^2]\xi_2^2 +$$
$$+ C\left[1 + \frac{C\dot\psi_0^2}{(A + B^\circ - C^\circ)\dot\psi_0^2 - (3g/R)(A + B^\circ - C - C^\circ)}\right]\xi_3^2 + 2C\dot\psi_0 m\eta\xi_3 -$$
$$- \left[(A + B^\circ - C^\circ)\dot\psi_0^2(n^2 - m^2) - Cr_0\dot\psi_0 n + Mg\,ln -\right.$$
$$\left.- \frac{3g}{R}(A + B^\circ - C - C^\circ)(n^2 - m^2)\right]\eta^2 + \ldots \text{const}$$

This integral becomes a positive-definite function of the arguments of η, ξ_1, ξ_2, ξ_3

$$(A+B^{\circ}-C^{\circ})\dot{\psi}_0^2(n^2-m^2)-Cr_0\dot{\psi}_0 n+Mgln-$$
$$-\frac{3g}{R}(A+B^{\circ}-C-C^{\circ})(n^2-m^2)<0 \qquad (3.4)$$

which on the strength of Liapunov's theorem is the sufficient condition for the stability of motion (3.1). In the case under consideration ($\sin\theta_0 \neq 0$) condition (3.4), by (3.3), can be transformed into

$$(A+B^{\circ}-C^{\circ})\dot{\psi}_0^2-\frac{3g}{R}(A+B^{\circ}-C-C^{\circ})>0 \qquad (3.5)$$

Thus, under conditions (3.5) the regular precession of a gyroscope on gimbals (balanced or unbalanced) is stable with respect to θ, $\dot{\theta}$, $\dot{\psi}$, r; hence it is also stable with respect to θ, $\dot{\theta}$, $\dot{\psi}$, $\dot{\phi}$.

We shall consider now the motion (3.1) when $\theta_0 = 0$, that is when the inner ring (the housing) rotates uniformly about the Oz_1-axis with angular velocity $\dot{\psi}_0$, and the gyroscope spins uniformly about the same axis with the angular velocity r_0. It is seen from (3.2) that in this case these constant angular velocities $\dot{\psi}_0$ and r_0 can be of any magnitude. The examination of the integral

$$V=V_1-2\dot{\psi}_0V_2+2C(\dot{\psi}_0-r_0)V_3$$

shows that in order to obtain the sufficient condition for stability of the motion considered, we must set $\theta_0 = 0$ in (3.4). In this way we have

$$(A+B^{\circ}-C^{\circ})\dot{\psi}_0^2-Cr_0\dot{\psi}_0+Mgl-\frac{3g}{R}(A+B^{\circ}-C-C^{\circ})<0 \qquad (3.6)$$

The above condition can be transformed and reduced to the form shown in [4].

All the conditions which we obtained can be reduced to the well-known sufficient conditions for stability of regular precession, or of vertical rotation of a heavy gyroscope on gimbals with the outer gimbal ring vertical, or to the motion of a heavy solid about a fixed point in the case of Lagrange. In order to obtain this reduction we reject terms containing R, since these terms characterize the non-parallel property of the force-field lines, or we set the moments of inertia of the gimbal rings equal to zero.

We shall demonstrate the necessity of condition (3.6). Let us consider first the function

$$V=(A+A^{\circ})\eta\dot{\eta}$$

and its time derivative, taken on the strength of the equations of the perturbed motion. This time derivative equals

$$\frac{dV}{dt} = (A + A^\circ)\,\dot{\eta}^2 + \Big[(A + B^\circ - C^\circ)\,\dot{\psi}_0{}^2 - Cr_0\dot{\psi}_0 + Mgl - \\ - \frac{3g}{R}(A + B^\circ - C - C^\circ)\Big]\,\eta^2 + \ldots$$

and when it satisfies in addition the condition

$$(A + B^\circ - C^\circ)\,\dot{\psi}_0{}^2 - Cr_0\dot{\psi}_0 + \mathrm{Mg}\,l - \frac{3g}{R}(A + B^\circ - C - C^\circ) > 0$$

then it becomes a positive-definite function of the variables η, $\dot{\eta}$, and the function V can assume positive values. Then, by Chetaev's theorem, the motion will be unstable. Thus, the condition (3.6) is (excepting the boundary) the necessary and sufficient condition for stability of the motion (3.1) when $\theta_0 = 0$.

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